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Special almost Hermitian geometry

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Abstract

We study the classification of special almost hermitian manifolds in Gray and Hervella's type classes. We prove that the exterior derivatives of the Kähler form and the complex volume form contain all the information about the intrinsic torsion of the $SU(n)$-structure. Furthermore, we apply the obtained results to almost hyperhermitian geometry. Thus, we show that the exterior derivatives of the three Kähler forms of an almost hyperhermitian manifold are sufficient to determine the three covariant derivatives of such forms, i.e., the three mentioned exterior derivatives determine the intrinsic torsion of the $Sp(n)$-structure.
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1. Introduction

In 1955, Berger [1] gave the list of possible holonomy groups of non-symmetric Riemannian *m*-manifolds whose holonomy representation is irreducible. Such a list of groups was complemented with their corresponding holonomy representations, i.e., it was also speci-

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fied the action of each group on the tangent space. Consequently, each group $G \subseteq SO(m)$ in Berger's list gives rise to a geometric structure. Moreover, the groups G may be given as the stabilisers in $SO(m)$ of certain differential forms on $\mathbb{R}^m$. For $G = G_2$, it is a three-form ϕ on $\mathbb{R}^7$; for $G = Spin(7)$, it is a four-form φ on $\mathbb{R}^8$; for $G = Sp(n)Sp(1)$, it is a four-form Ω on $\mathbb{R}^{4n}$; for $G = U(n)$, a Kähler form ω on $\mathbb{R}^{2n}$, etc. Such forms are a key ingredient in the definition of the corresponding G-structure on a Riemannian m-manifold M. Furthermore, the intrinsic torsion of a G-structure, defined in next section, can be identified with the Levi-Civita covariant derivatives of the corresponding forms and is always contained in $\mathcal{W} = T^*M \otimes \mathfrak{g}^\perp$, being $\mathfrak{so}(m) = \mathfrak{g} \oplus \mathfrak{g}^\perp$. The action of G splits $\mathcal{W}$ into irreducible components, say $\mathcal{W} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_k$. Then, G-structures on M can be classified in at most 2^k classes.

This way of classifying G-structures was initiated by Gray and Hervella [8], where they considered the case $G = U(n)$ (almost Hermitian structures), turning out $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, for $n > 2$, i.e., there are sixteen classes of almost Hermitian manifolds. Later, diverse authors have studied the situation for other G-structures: G_2, $Spin(7)$, $Sp(n)Sp(1)$, etc.

In the present paper we study the situation for $G = SU(n)$. Thus, we consider Riemannian $2n$-manifolds equipped with a Kähler form ω and a complex volume form $\Psi = \psi_+ + i\psi_-$, called special almost Hermitian manifolds. The group $SU(n)$ is the stabiliser in $SO(2n)$ of ω and Ψ. Therefore, the information about intrinsic torsion of an $SU(n)$-structure is contained in $\nabla\omega$ and $\nabla\Psi$, where ∇ denotes the Levi-Civita connection. For high dimensions, $2n \geq 8$, we find

$$T^*M \otimes \mathfrak{su}(n)^\perp = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5,$$

where the first four summands coincide with Gray and Hervella's ones and $\mathcal{W}_5 \cong T^*M$. Besides the additional summand $\mathcal{W}_5$, another interesting difference may be pointed out: all the information about the torsion of the $SU(n)$-structure, $n \geq 4$, is contained in the exterior derivatives $d\omega$ and $d\psi_+$, or $d\omega$ and $d\psi_-$. This happens similarly for another G-structures, $d\varphi$ is sufficient to classify a $Spin(7)$-structure, $d\Omega$ is sufficient to know the intrinsic $Sp(n)Sp(1)$-torsion, $n > 2$, etc. However, we recall that $d\omega$ is not enough to classify a $U(n)$-structure, we also need to search in the Nijenhuis tensor for the remaining information. Moreover, the importance of $SU(n)$-structures from the point of view of geometry and theoretical physics makes valuable a detailed description of the involved tensors $\nabla\omega$ and $\nabla\Psi$. Here we describe $\nabla\Psi$ which complements the study of $\nabla\omega$ done by Gray and Hervella.

The paper is organised as follows. In Section 2, we start discussing basic results. Then we pay attention to the study of special almost hermitian $2n$-manifolds of high dimensions, $2n \geq 8$. However, some results involving the cases $n = 2, 3$ are also given. For instance, for $n \geq 2$, we prove the invariance under conformal changes of metric of a certain one-form related with parts $\mathcal{W}_4$ and $\mathcal{W}_5$ of the intrinsic torsion. This is a generalization of a Chiossi and Salamon's result for $SU(3)$-structures [3].

In Section 3, we study special almost Hermitian manifold of low dimensions. Such manifolds of six dimensions have been studied in [3]. Here we show some additional detailed information. When $n = 1, 2, 3$, the number of special peculiarities that occur is big enough to justify a separated exposition. In particular, we prove that, for these manifolds, $d\omega$, $d\psi_+$ *and* $d\psi_-$ are sufficient to know the intrinsic torsion.

Finally, as examples of $SU(2n)$-structures, we consider almost hyperhermitian manifolds in Section 4. We show that the exterior derivatives $d\omega_I$, $d\omega_J$ and $d\omega_K$ of the Kähler forms are enough to compute the covariant derivatives $\nabla\omega_I$, $\nabla\omega_J$ and $\nabla\omega_K$. This implies Hitchin's result [9] that if ω_I, ω_J and ω_K are closed, then they are covariant constant, i.e., the manifold is hyperkähler. Furthermore, we prove that locally conformal hyperkähler manifolds are equipped with three $SU(2n)$-structures of type $\mathcal{W}_4 \oplus \mathcal{W}_5$, respectively associated with the almost complex structures I, J and K. As a consequence of this result, we obtain an alternative proof of the Ricci flatness of the metric of hyperkähler manifolds.

2. Special almost Hermitian manifolds

An *almost Hermitian* manifold is a $2n$-dimensional manifold M, $n > 0$, with a $U(n)$-structure. This means that M is equipped with a Riemannian metric $\langle\cdot,\cdot\rangle$ and an orthogonal almost complex structure I. Each fibre T_mM of the tangent bundle can be consider as complex vector space by defining $ix = Ix$. We will write $T_mM_{\mathbb{C}}$ when we are regarding T_mM as such a space.

We define a Hermitian scalar product $\langle\cdot,\cdot\rangle_{\mathbb{C}} = \langle\cdot,\cdot\rangle + \mathrm{i}\omega(\cdot,\cdot)$, where ω is the Kähler form given by $\omega(x, y) = \langle x, Iy\rangle$. The real tangent bundle TM is identified with the cotangent bundle T^*M by the map $x \to \langle\cdot, x\rangle = x$. Analogously, the conjugate complex vector space $\overline{T_mM_{\mathbb{C}}}$ is identified with the dual complex space $T_m^*M_{\mathbb{C}}$ by the map $x \to \langle\cdot, x\rangle_{\mathbb{C}} = x_{\mathbb{C}}$. It follows immediately that $x_{\mathbb{C}} = x + \mathrm{i}Ix$.

If we consider the spaces $\Lambda^pT_m^*M_{\mathbb{C}}$ of skew-symmetric complex forms, one can check $x_{\mathbb{C}} \wedge y_{\mathbb{C}} = (x + \mathrm{i}Ix) \wedge (y + \mathrm{i}Iy)$. There are natural extensions of scalar products to $\Lambda^pT_m^*M$ and $\Lambda^pT_m^*M_{\mathbb{C}}$, respectively defined by

$$\langle a, b\rangle = \frac{1}{p!}\sum_{i_1,\dots,i_p=1}^{2n} a(e_{i_1},\dots,e_{i_p})b(e_{i_1},\dots,e_{i_p}),$$

$$\langle a_{\mathbb{C}}, b_{\mathbb{C}}\rangle_{\mathbb{C}} = \frac{1}{p!}\sum_{i_1,\dots,i_p=1}^{n} a_{\mathbb{C}}(u_{i_1},\dots,u_{i_p})\overline{b_{\mathbb{C}}(u_{i_1},\dots,u_{i_p})},$$

where $e_1,\dots,e_{2n}$ is an orthonormal basis for real vectors and $u_1,\dots,u_n$ is a unitary basis for complex vectors.

The following conventions will be used in this paper. If b is a $(0, s)$-tensor, we write

$$I_{(i)}b(X_1,\dots,X_i,\dots,X_s) = -b(X_1,\dots,IX_i,\dots,X_s),$$

$$Ib(X_1,\dots,X_s) = (-1)^s b(IX_1,\dots,IX_s), \quad i_Ib = (I_{(1)} + \cdots + I_{(s)})b,$$

$$L(b) = \sum_{1\le i<j\le s} I_{(i)}I_{(j)}b, \quad s \ge 2. \tag{2.1}$$

A *special almost Hermitian* manifold is a $2n$-dimensional manifold M with an $SU(n)$-structure. This means that $(M, \langle\cdot,\cdot\rangle, I)$ is an almost Hermitian manifold equipped with a complex volume form $\Psi = \psi_+ + \mathrm{i}\psi_-$ such that $\langle\Psi, \Psi\rangle_{\mathbb{C}} = 1$. Note that $I_{(i)}\psi_+ = \psi_-$.

If $e_1, \dots, e_n$ is a unitary basis for complex vectors such that $\Psi(e_1, \dots, e_n) = 1$, i.e., $\psi_+(e_1, \dots, e_n) = 1$ and $\psi_-(e_1, \dots, e_n) = 0$, then $e_1, \dots, e_n, Ie_1, \dots, Ie_n$ is an orthonormal basis for real vectors *adapted* to the $SU(n)$-structure. Furthermore, if A is a matrix relating two adapted basis of an $SU(n)$-structure, then $A \in SU(n) \subseteq SO(2n)$. On the other hand, it is straightforward to check

$$\omega^n = (-1)^{n(n+1)/2} n!\, e_1 \wedge \cdots \wedge e_n \wedge Ie_1 \wedge \cdots \wedge Ie_n,$$

where $\omega^n = \omega \wedge \dots^{(n)} \wedge \omega$.

If we fix the form *Vol* such that $(-1)^{n(n+1)/2} n!\, Vol = \omega^n$ as real volume form, it follows next lemma.

Lemma 2.1. *Let M be a special almost Hermitian* $2n$*-manifold, then*

(i) $\psi_+ \wedge \omega = \psi_- \wedge \omega = 0$;
(ii) *for n odd, we have* $\psi_+ \wedge \psi_- = -(-1)^{n(n+1)/2} 2^{n-1}\, Vol$ *and* $\psi_+ \wedge \psi_+ = \psi_- \wedge \psi_- = 0$;
(iii) *for n even, we have* $\psi_+ \wedge \psi_+ = \psi_- \wedge \psi_- = (-1)^{n(n+1)/2} 2^{n-1}\, Vol$ *and* $\psi_+ \wedge \psi_- = 0$;
(iv) *for* $n \geq 2$ *and* $1 \leq i < j \leq n$, $I_{(i)} I_{(j)} \psi_+ = -\psi_+$ *and* $I_{(i)} I_{(j)} \psi_- = -\psi_-$; *and*
(v) $x \wedge \psi_+ = Ix \wedge \psi_- = -(Ix \lrcorner \psi_+) \wedge \omega$ *and* $x \lrcorner \psi_+ = Ix \lrcorner \psi_-$, *for all vector x, where* $\lrcorner$ *denotes the interior product.*

Proof. All parts follow by a straightforward way, taking the identities

$$\psi_+ = Re(e_{1\mathbb{C}} \wedge \cdots \wedge e_{n\mathbb{C}}), \qquad \psi_- = Im(e_{1\mathbb{C}} \wedge \cdots \wedge e_{n\mathbb{C}}), \tag{2.2}$$

$$\omega = \sum_{i=1}^{n} Ie_i \wedge e_i, \tag{2.3}$$

into account, where $e_1, \dots, e_n, Ie_1, \dots, Ie_n$ is an adapted basis to the $SU(n)$-structure. Note that parts (ii) and (iii) can be given together by the equation

$$n! \Psi \wedge \bar{\Psi} = i^n (-1)^{n(n-1)/2} 2^n \omega^n. \quad \square$$

We will also need to consider the contraction of a p-form b by a skew-symmetric contravariant two-vector $x \wedge y$, i.e., $(x \wedge y) \lrcorner b(x_1, \dots, x_{p-2}) = b(x, y, x_1, \dots, x_{p-2})$. When $n \geq 2$, it is obvious that $(Ix \wedge y) \lrcorner \psi_+ = -(x \wedge y) \lrcorner \psi_-$. Furthermore, let us note that there are two Hodge star operators defined on M. Such operators, denoted by $*$ and $*_{\mathbb{C}}$, are respectively associated with the volume forms *Vol* and Ψ.

Relative to the real Hodge star operator, we have the following results.

Lemma 2.2. *For any one-form* μ *we have*

$$*(*(\mu \wedge \psi_+) \wedge \psi_+) = *(*(\mu \wedge \psi_-) \wedge \psi_-) = -2^{n-2} \mu,$$
$$*(*(\mu \wedge \psi_-) \wedge \psi_+) = - * (*(\mu \wedge \psi_+) \wedge \psi_-) = 2^{n-2} I\mu.$$

Proof. The identities follow by direct computation, taking Eq (2.2) into account. □

We are dealing with G-structures where G is a subgroup of the linear group $GL(m, \mathbb{R})$. If M possesses a G-structure, then there always exists a G-connection defined on M. Moreover, if $(M^m, \langle\cdot,\cdot\rangle)$ is an orientable m-dimensional Riemannian manifold and G is a closed and connected subgroup of $SO(m)$, then there exists a unique metric G-connection $\tilde{\nabla}$ such that $\xi_x = \tilde{\nabla}_x - \nabla_x$ takes its values in $\mathfrak{g}^\perp$, where $\mathfrak{g}^\perp$ denotes the orthogonal complement in $\mathfrak{so}(m)$ of the Lie algebra $\mathfrak{g}$ of G and ∇ denotes the Levi-Civita connection [13,4]. The tensor ξ is the *intrinsic torsion* of the G-structure and $\tilde{\nabla}$ is called the *minimal G-connection*.

For $U(n)$-structures, the minimal $U(n)$-connection is given by $\tilde{\nabla} = \nabla + \xi$, with

$$\xi_X Y = -\tfrac{1}{2} I(\nabla_X I)Y. \tag{2.4}$$

see [5]. Since $U(n)$ stabilises the Kähler form ω, it follows that $\tilde{\nabla}\omega = 0$. Moreover, the equation $\xi_X(IY) + I(\xi_X Y) = 0$ implies $\nabla\omega = -\xi\omega \in T^*M \otimes \mathfrak{u}(n)^\perp$. Thus, one can identify the $U(n)$-components of ξ with the $U(n)$-components of $\nabla\omega$.

For $SU(n)$-structures, we have the decomposition $\mathfrak{so}(2n) = \mathfrak{su}(n) + \mathbb{R} + \mathfrak{u}(n)^\perp$, i.e., $\mathfrak{su}(n)^\perp = \mathbb{R} + \mathfrak{u}(n)^\perp$. Therefore, the intrinsic $SU(n)$-torsion $\eta + \xi$ is such that $\eta \in T^*M \otimes \mathbb{R} \cong T^*M$ and ξ is still determined by Eq. (2.4). The tensors ω, ψ_+ and ψ_- are stabilised by the $SU(n)$-action, and $\bar{\nabla}\omega = 0$, $\bar{\nabla}\psi_+ = 0$ and $\bar{\nabla}\psi_- = 0$, where $\bar{\nabla} = \nabla + \eta + \xi$ is the minimal $SU(n)$-connection. Since $\bar{\nabla}$ is metric and $\eta \in T^*M \otimes \mathbb{R}$, we have $\langle Y, \eta_X Z\rangle = (I\eta)(X)\omega(Y, Z)$, where η on the right side is a one-form. Hence

$$\eta_X Y = I\eta(X)IY. \tag{2.5}$$

We can check $\eta\omega = 0$, then from $\bar{\nabla}\omega = 0$ we obtain:

(i) for $n = 1$, $\nabla\omega = -\xi\omega \in T^*M \otimes \mathfrak{u}(1)^\perp = \{0\}$;
(ii) for $n = 2$, $\nabla\omega = -\xi\omega \in T^*M \otimes \mathfrak{u}(2)^\perp = \mathcal{W}_2 + \mathcal{W}_4$;
(iii) for $n \geq 3$, $\nabla\omega = -\xi\omega \in T^*M \otimes \mathfrak{u}(n)^\perp = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4$;

where the summands $\mathcal{W}_i$ are the irreducible $U(n)$-modules given by Gray and Hervella [8] and + denotes direct sum. In general, these spaces $\mathcal{W}_i$ are also irreducible as $SU(n)$-modules. The only exceptions are $\mathcal{W}_1$ and $\mathcal{W}_2$ when $n = 3$. In fact, for that case, we have the following decompositions into irreducible $SU(3)$-components,

$$\mathcal{W}_i = \mathcal{W}_i^+ + \mathcal{W}_i^-, \quad i = 1, 2,$$

where the space $\mathcal{W}_i^+$ ($\mathcal{W}_i^-$) consists in those tensors $a \in \mathcal{W}_i \subseteq T^*M \otimes \Lambda^2 T^*M$ such that the bilinear form $r(a)$, defined by $2r(a) = \langle x\lrcorner\psi_+, y\lrcorner a\rangle$, is symmetric (skew-symmetric).

On the other hand, since $\bar{\nabla}\psi_+ = 0$ and $\bar{\nabla}\psi_- = 0$, we have $\nabla\psi_+ = -\eta\psi_+ - \xi\psi_+$ and $\nabla\psi_- = -\eta\psi_- - \xi\psi_-$. Therefore, from Eqs. (2.4) and (2.5) we obtain the following expressions

$$\begin{aligned} -\eta_X\psi_+ &= -nI\eta(X)\psi_-, & -\xi_X\psi_+ &= \tfrac{1}{2}(e_i\lrcorner\nabla_X\omega) \wedge (e_i\lrcorner\psi_-), \\ -\eta_X\psi_- &= nI\eta(X)\psi_+, & -\xi_X\psi_- &= -\tfrac{1}{2}(e_i\lrcorner\nabla_X\omega) \wedge (e_i\lrcorner\psi_+), \end{aligned} \tag{2.6}$$

where the summation convention is used.

It is obvious that $-\eta\psi_+ \in \mathcal{W}_5^- = T^*M \otimes \psi_-$ and $-\eta\psi_- \in \mathcal{W}_5^+ = T^*M \otimes \psi_+$. The tensors $-\xi\psi_+$ and $-\xi\psi_-$ are described in the following proposition, where we need to consider the two $SU(n)$-maps

$$\Xi_+, \Xi_- : T^*M \otimes \mathfrak{u}(n)^\perp \to T^*M \otimes \Lambda^n T^*M$$

respectively defined by $\nabla \cdot \omega \to 1/2\,(e_i \lrcorner \nabla \cdot \omega) \wedge (e_i \lrcorner \psi_-)$ and $\nabla \cdot \omega \to -1/2\,(e_i \lrcorner \nabla \cdot \omega) \wedge (e_i \lrcorner \psi_+)$. Likewise, we also consider the $SU(n)$-spaces $[\![\lambda^{p,0}]\!] = \{Re(b_{\mathbb{C}}) \,|\, b_{\mathbb{C}} \in \Lambda^p T^*M_{\mathbb{C}}\}$ of real p-forms. Thus, $[\![\lambda^{0,0}]\!] = \mathbb{R}$, $[\![\lambda^{1,0}]\!] = T^*M$ and, for $p \geq 2$, $[\![\lambda^{p,0}]\!] = \{b \in \Lambda^p T^*M \,|\, I_{(i)} I_{(j)} b = -b,\ 1 \leq i < j \leq p\}$. We write $[\![\lambda^{p,0}]\!]$ in agreeing with notations used in [13,5].

Proposition 2.3. *For $n \geq 3$, the $SU(n)$-maps Ξ_+ and Ξ_- are injective and*

$$\Xi_+(T^*M \otimes \mathfrak{u}(n)^\perp) = \Xi_-(T^*M \otimes \mathfrak{u}(n)^\perp) = T^*M \otimes [\![\lambda^{n-2,0}]\!] \wedge \omega.$$

For $n = 2$, the maps Ξ_+ and Ξ_- are not injective, being

$$\begin{aligned} &ker\ \Xi_+ = T^*M \otimes \psi_-, \qquad ker\ \Xi_- = T^*M \otimes \psi_+, \\ &\Xi_+(T^*M \otimes \mathfrak{u}(2)^\perp) = \Xi_-(T^*M \otimes \mathfrak{u}(2)^\perp) = T^*M \otimes \omega. \end{aligned}$$

Proof. We consider $n \geq 2$. As the real metric $\langle \cdot, \cdot \rangle$ is Hermitian with respect to I, we have $I(\nabla_X \omega) = -\nabla_X \omega$ [8], for all vector X. But this is equivalent to

$$\nabla_X \omega = \sum_{1 \leq i < j \leq n}^{n} (a_{ij} \mathrm{Re}(e_{i\mathbb{C}} \wedge e_{j\mathbb{C}}) + b_{ij} \mathrm{Im}(e_{i\mathbb{C}} \wedge e_{j\mathbb{C}})) \in [\![\lambda^{2,0}]\!],$$

where $e_1, \ldots, e_n, Ie_1, \ldots, Ie_n$ is an adapted basis. Taking (2.6) into account, it is straightforward to check

$$\begin{aligned} \Xi_+(\nabla_X \omega) = &- \sum_{1 \leq i < j \leq n}^{n} a_{ij} \mathrm{Re}(*_{\mathbb{C}}(e_{i\mathbb{C}} \wedge e_{j\mathbb{C}})) \wedge \omega \\ &+ \sum_{1 \leq i < j \leq n}^{n} b_{ij} \mathrm{Im}(*_{\mathbb{C}}(e_{i\mathbb{C}} \wedge e_{j\mathbb{C}})) \wedge \omega \in [\![\lambda^{2,0}]\!] \wedge \omega, \\ \Xi_-(\nabla_X \omega) = &- \sum_{1 \leq i < j \leq n}^{n} a_{ij} \mathrm{Im}(*_{\mathbb{C}}(e_{i\mathbb{C}} \wedge e_{j\mathbb{C}})) \wedge \omega \\ &- \sum_{1 \leq i < j \leq n}^{n} b_{ij} \mathrm{Re}(*_{\mathbb{C}}(e_{i\mathbb{C}} \wedge e_{j\mathbb{C}})) \wedge \omega \in [\![\lambda^{2,0}]\!] \wedge \omega. \end{aligned}$$

From these equations Proposition follows. □

For sake of simplicity, for $n \geq 2$, we denote $\mathcal{W}^{\Xi} = T^*M \otimes [\![\lambda^{n-2,0}]\!] \wedge \omega$. Moreover, we will consider the map $\mathcal{L} : T^*M \otimes \Lambda^n T^*M \to T^*M \otimes \Lambda^n T^*M$ defined by

$$\mathcal{L}(b) = I_{(1)}(I_{(2)} + \cdots + I_{(n+1)})b. \tag{2.7}$$

Proposition 2.3 and above considerations give rise to the following theorem where we describe the properties satisfied by the $SU(n)$-components of $\nabla\psi_+$ and $\nabla\psi_-$.

Theorem 2.4. *Let M be a special almost Hermitian $2n$-manifold, $n \geq 4$, with Kähler form ω and complex volume form $\Psi = \psi_+ + \mathrm{i}\psi_-$. Then*

$$\nabla\psi_+ \in \mathcal{W}_1^{\Xi} + \mathcal{W}_2^{\Xi} + \mathcal{W}_3^{\Xi} + \mathcal{W}_4^{\Xi} + \mathcal{W}_5^{-},$$
$$\nabla\psi_- \in \mathcal{W}_1^{\Xi} + \mathcal{W}_2^{\Xi} + \mathcal{W}_3^{\Xi} + \mathcal{W}_4^{\Xi} + \mathcal{W}_5^{+},$$

where $\mathcal{W}_i^{\Xi} = \Xi_+(\mathcal{W}_i) = \Xi_-(\mathcal{W}_i)$, $\mathcal{W}_5^{+} = T^*M \otimes \psi_+$ and $\mathcal{W}_5^{-} = T^*M \otimes \psi_-$. The modules $\mathcal{W}_i^{\Xi}$ are explicitly described by

$$\mathcal{W}_1^{\Xi} = \{e_i \otimes Ie_i \wedge b \wedge \omega + e_i \otimes e_i \wedge I_{(1)}b \wedge \omega \mid b \in [\![\lambda^{n-3,0}]\!]\},$$
$$\mathcal{W}_2^{\Xi} = \{b \in \mathcal{W}^{\Xi} \mid \mathcal{L}(b) = (n-2)b \text{ and } \tilde{a}(b) \wedge \omega = 0\},$$
$$\mathcal{W}_1^{\Xi} + \mathcal{W}_2^{\Xi} = \{b \in \mathcal{W}^{\Xi} \mid \mathcal{L}(b) = (n-2)b\},$$
$$\mathcal{W}_3^{\Xi} = \{b \in \mathcal{W}^{\Xi} \mid \tilde{a}(b) = 0\},$$
$$\mathcal{W}_4^{\Xi} = \{e_i \otimes ((x \wedge e_i) \lrcorner \psi_+) \wedge \omega \mid x \in TM\} = \{e_i \otimes ((x \wedge e_i) \lrcorner \psi_-) \wedge \omega \mid x \in TM\},$$
$$\mathcal{W}_3^{\Xi} + \mathcal{W}_4^{\Xi} = \{b \in \mathcal{W}^{\Xi} \mid \mathcal{L}(b) = -(n-2)b\},$$

where $\tilde{a}$ denotes the alternation map.

Proof. Some parts of Theorem follow by computing the image $\Xi_+(\nabla\omega)_i$ of the $\mathcal{W}_i$-part of $\nabla\omega$, taking the properties for $\mathcal{W}_i$ given in [8] into account, and others, with Schur's Lemma [2] in mind, by computing $\Xi_+(a)$, where $0 \neq a \in \mathcal{W}_i$. □

If we consider the alternation maps $\tilde{a}_{\pm} : \mathcal{W}^{\Xi} + \mathcal{W}_5^{\mp} \to \Lambda^{n+1}T^*M$, we get the following consequences of Theorem 2.4.

Corollary 2.5. *For $n \geq 4$, the exterior derivatives of ψ_+ and ψ_- are such that*

$$\mathrm{d}\psi_+, \mathrm{d}\psi_- \in T^*M \wedge [\![\lambda^{n-2,0}]\!] \wedge \omega = \mathcal{W}_1^{a} + \mathcal{W}_2^{a} + \mathcal{W}_{4,5}^{a},$$

where $\tilde{a}_{\pm}(\mathcal{W}_1^{\Xi}) = \mathcal{W}_1^{a}$, $\tilde{a}_{\pm}(\mathcal{W}_2^{\Xi}) = \mathcal{W}_2^{a}$ and $\tilde{a}_{\pm}(\mathcal{W}_4^{\Xi}) = \tilde{a}_{\pm}(\mathcal{W}_5^{\mp}) = \mathcal{W}_{4,5}^{a}$. Moreover, $ker(\tilde{a}_{\pm}) = \mathcal{W}_3^{\Xi} + \mathcal{A}_{\pm}$, where $T^*M \cong \mathcal{A}_{\pm} \subseteq \mathcal{W}_4^{\Xi} + \mathcal{W}_5^{\mp}$, and the modules $\mathcal{W}_i^{a}$ are described by

$$\mathcal{W}_1^{a} = [\Lambda^{n-3,0}] \wedge \omega \wedge \omega,$$
$$\mathcal{W}_2^{a} = \{b \in T^*M \wedge [\![\lambda^{n-2,0}]\!] \wedge \omega \mid b \wedge \omega = 0 \text{ and } *b \wedge \psi_+ = 0\}$$
$$= \{b \in T^*M \wedge [\![\lambda^{n-2,0}]\!] \wedge \omega \mid b \wedge \omega = 0 \text{ and } *b \wedge \psi_- = 0\},$$
$$\mathcal{W}_{4,5}^{a} = T^*M \wedge \psi_+ = T^*M \wedge \psi_- = [\![\lambda^{n-1,0}]\!] \wedge \omega.$$

Note also that

$$\mathcal{W}_1^a + \mathcal{W}_2^a = \{b \in T^*M \wedge [\![\lambda^{n-2,0}]\!] \wedge \omega \mid *b \wedge \psi_+ = 0\}$$
$$= \{b \in T^*M \wedge [\![\lambda^{n-2,0}]\!] \wedge \omega \mid *b \wedge \psi_- = 0\},$$
$$\mathcal{W}_2^a + \mathcal{W}_{4,5}^a = \{b \in T^*M \wedge [\![\lambda^{n-2,0}]\!] \wedge \omega \mid b \wedge \omega = 0\}.$$

In this point we already have all the ingredients to explicitly describe the one-form η. This will complete the definition of the $SU(n)$-connection $\bar{\nabla}$.

Theorem 2.6. *For an $SU(n)$-structure, $n \geq 2$, the $\mathcal{W}_5$-part η of the torsion can be identified with $-\eta\psi_+ = -nI\eta \otimes \psi_-$ or $-\eta\psi_- = nI\eta \otimes \psi_+$, where η is a one-form such that*

$$*(*d\psi_+ \wedge \psi_+ + *d\psi_- \wedge \psi_-) = n2^{n-1}\eta + 2^{n-2}I\mathrm{d}^*\omega,$$

or

$$*(*\mathrm{d}\psi_+ \wedge \psi_- - *\mathrm{d}\psi_- \wedge \psi_+) = n2^{n-1}I\eta - 2^{n-2}\mathrm{d}^*\omega.$$

Furthermore, if $n \geq 3$, then $*\mathrm{d}\psi_+ \wedge \psi_+ = *\mathrm{d}\psi_- \wedge \psi_-$ and *$*\mathrm{d}\psi_+ \wedge \psi_- = -*\mathrm{d}\psi_- \wedge \psi_+$.*

Proof. We prove the result for $n \geq 4$ and we will see the cases $n = 2, 3$ in next section. The $\mathcal{W}_4$-part of $\nabla\omega$ is given by $2(n-1)(\nabla\omega)_4 = e_i \otimes e_i \wedge \mathrm{d}^*\omega + e_i \otimes Ie_i \wedge I\mathrm{d}^*\omega$ [8]. Then, by computing $\Xi_+(\nabla\omega)_4$, we get

$$(\nabla\psi_+)_4 = -\frac{1}{2(n-1)} e_i \otimes ((\mathrm{d}^*\omega \wedge e_i) \lrcorner \psi_+) \wedge \omega. \tag{2.8}$$

Now, since $(\nabla\psi_+)_5 = -nI\eta \otimes \psi_-$, we have

$$\tilde{a}_+((\nabla\psi_+)_4 + (\nabla\psi_+)_5) = -\tfrac{1}{2}(\mathrm{d}^*\omega \lrcorner \psi_+) \wedge \omega - nI\eta \wedge \psi_- = -(\tfrac{1}{2}I\mathrm{d}^*\omega + n\eta) \wedge \psi_+.$$

Hence, the $\mathcal{W}_{4,5}^a$-part of $\mathrm{d}\psi_+$ is given by

$$(\mathrm{d}\psi_+)_{4,5} = -(n\eta + \tfrac{1}{2}I\mathrm{d}^*\omega) \wedge \psi_+, \tag{2.9}$$

Finally, taking Lemma 2.2 into account, it follows

$$*(*\mathrm{d}\psi_+ \wedge \psi_+) = *((\mathrm{d}\psi_+)_{4,5} \wedge \psi_+) = n2^{n-2}\eta + 2^{n-3}I\mathrm{d}^*\omega,$$
$$*(*\mathrm{d}\psi_+ \wedge \psi_-) = *((\mathrm{d}\psi_+)_{4,5} \wedge \psi_-) = n2^{n-2}I\eta - 2^{n-3}\mathrm{d}^*\omega.$$

The identities for $\mathrm{d}\psi_-$ can be proved in a similar way. □

Remark 2.7.

(i) It is known that $I\mathrm{d}^*\omega = *(*\mathrm{d}\omega \wedge \omega) = -\langle \cdot \lrcorner \mathrm{d}\omega, \omega\rangle$. Therefore, Theorem 2.6 says that, for $n \geq 3$, η can be computed from $\mathrm{d}\omega$ and $\mathrm{d}\psi_+$ (or $\mathrm{d}\psi_-$). For $n = 2$, we will need $\mathrm{d}\omega$, $\mathrm{d}\psi_+$ *and* $\mathrm{d}\psi_-$ to determine the one-form η.

(ii) From Eq. (2.9), it follows that $\mathcal{A}_+ \subseteq ker(\tilde{a}_+)$ is given by

$$\mathcal{A}_+ = \left\{ -\frac{1}{2(n-1)} e_i \otimes ((x \wedge e_i) \lrcorner \psi_+) \wedge \omega - \frac{1}{2} x \otimes \psi_- \mid x \in TM \right\}.$$

Analogously, for $\mathcal{A}_- \subseteq ker(\tilde{a}_-)$, we have

$$\mathcal{A}_- = \left\{ -\frac{1}{2(n-1)} e_i \otimes ((x \wedge e_i) \lrcorner \psi_-) \wedge \omega + \frac{1}{2} x \otimes \psi_+ \mid x \in TM \right\}.$$

Since $\mathrm{d}\omega \in \mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}_4$ and $\mathrm{d}\psi_+, d\psi_- \in \mathcal{W}_1^a + \mathcal{W}_2^a + \mathcal{W}_{4,5}^a$, all the information about the intrinsic torsion of an $SU(n)$-structure, $n \geq 4$, is contained in $\mathrm{d}\omega$ and $\mathrm{d}\psi_+$ (or $\mathrm{d}\psi_-$). We recall that, for a $U(n)$-structure, $n \geq 2$, we need the Nijenhuis tensor and $\mathrm{d}\omega$ to have the complete information about the intrinsic torsion. Eq. (2.8) and Theorem 2.6 give us the components $\mathcal{W}_4$ and $\mathcal{W}_5$ of $\nabla\psi_+$ in terms of $\mathrm{d}\omega$ and $\mathrm{d}\psi_+$. For sake of completeness, we will compute the remaining parts of $\nabla\psi_+$ in terms of $\mathrm{d}\omega$ and $\mathrm{d}\psi_+$. To achieve this, let us study the behavior of the coderivatives $\mathrm{d}^*\psi_+$, $\mathrm{d}^*\psi_-$ and the forms $\mathrm{d}^*_\omega\psi_+$ and $\mathrm{d}^*_\omega\psi_-$ respectively defined by the contraction of $\nabla\psi_+$ and $\nabla\psi_-$ by ω, i.e.,

$$\mathrm{d}^*_\omega\psi_+(Y_1, \ldots, Y_{n-1}) = \nabla_{e_i}\psi_+(Ie_i, Y_1, \ldots, Y_{n-1})$$

and an analog expression gives $\mathrm{d}^*_\omega\psi_-$.

Note that $\mathrm{d}^*\psi_+ = - * d * \psi_+$ and $\mathrm{d}^*\psi_- = - * d * \psi_-$. By Lemma 2.1, when n is odd (even), $*\psi_+ = -(-1)^{n(n+1)/2}\psi_-$ and $*\psi_- = (-1)^{n(n+1)/2}\psi_+$ ($*\psi_+ = (-1)^{n(n+1)/2}\psi_+$ and $*\psi_- = (-1)^{n(n+1)/2}\psi_-$). Therefore, by Corollary 2.5, it is immediate that

$$\mathrm{d}^*\psi_+, \mathrm{d}^*\psi_- \in *(T^*M \wedge [\![\lambda^{n-2,0}]\!] \wedge \omega) = \mathcal{W}_1^c + \mathcal{W}_2^c + \mathcal{W}_{4,5}^c,$$

where the modules $\mathcal{W}_i^c$ are described in the following lemma.

Lemma 2.8. *For $n \geq 4$, $\mathcal{W}^c = *(T^*M \wedge [\![\lambda^{n-2,0}]\!] \wedge \omega)$ and L the map defined by (2.1), the modules $\mathcal{W}_1^c$, $\mathcal{W}_2^c$* and *$\mathcal{W}_{4,5}^c$ are defined by:*

$$\begin{aligned}
&\mathcal{W}_1^c = [\![\lambda^{n-3,0}]\!] \wedge \omega,\\
&\mathcal{W}_2^c = \{a \in \mathcal{W}^c \mid a \wedge \omega \wedge \omega = 0 \, and \, a \wedge \psi_+ = 0\},\\
&\mathcal{W}_1^c + \mathcal{W}_2^c = \{a \in \mathcal{W}^c \mid -2L(a) = (n-2)(n-5)a\} = \{a \in \mathcal{W}^c \mid a \wedge \psi_+ = 0\},\\
&\mathcal{W}_{4,5}^c = [\Lambda^{n-1,0}] = \{x \lrcorner \psi_+ \mid x \in TM\},\\
&\mathcal{W}_2^c + \mathcal{W}_{4,5}^c = \{a \in \mathcal{W}^c \mid a \wedge \omega \wedge \omega = 0\}.
\end{aligned}$$

Proof. It follows by applying $*$ to the $\mathcal{W}_i^a$ modules of Corollary 2.5.

For the description of $\mathcal{W}_1^c + \mathcal{W}_2^c$ involving the map L. Taking Proposition 2.3 into account, we consider $\nabla\psi_+ = x \otimes b \wedge \omega \in T^*M \otimes [\![\lambda^{n-2,0}]\!] \wedge \omega$. Now, making use of

Theorem 2.4, we obtain the $\mathcal{W}_1^{\Xi} + \mathcal{W}_2^{\Xi}$-part of $\nabla\psi_+$,

$$(\nabla\psi_+)_{1,2} = \frac{(n-2)\nabla\psi_+ + \mathcal{L}(\nabla\psi_+)}{2(n-2)} = \frac{1}{2}(x \otimes b \wedge \omega + Ix \otimes I_{(1)}b \wedge \omega). \tag{2.10}$$

Then, we compute $(\mathrm{d}^*\psi_+)_{1,2} = \mathrm{d}^*(\nabla\psi_+)_{1,2}$ and check $-2L(\mathrm{d}^*\psi_+)_{1,2} = (n-2)(n-5)(\mathrm{d}^*\psi_+)_{1,2}$. □

In the following result, $(\mathrm{d}^*\psi_+)_i$ and $(\mathrm{d}^*_\omega\psi_+)_i$ ($(\mathrm{d}^*\psi_-)_i$ and $(\mathrm{d}^*_\omega\psi_-)_i$) respectively denote the images by the maps d^* and d^*_ω of the $\mathcal{W}_i$-component of $\nabla\psi_+$ ($\nabla\psi_-$). This notation for the $\mathcal{W}_i$-part of a tensor will be used in the following.

Lemma 2.9. *For $n \geq 3$ and the map i_I given by (2.1), the forms $\mathrm{d}^*\psi_+$, $\mathrm{d}^*\psi_-$, $\mathrm{d}^*_\omega\psi_+$ and $\mathrm{d}^*_\omega\psi_+$ satisfy*:

$$\begin{aligned}
&(\mathrm{d}^*\psi_+)_{1,2} = -(\mathrm{d}^*_\omega\psi_-)_{1,2}, && (\mathrm{d}^*_\omega\psi_+)_{1,2} = (\mathrm{d}^*\psi_-)_{1,2},\\
&(\mathrm{d}^*\psi_+)_4 = (\mathrm{d}^*_\omega\psi_-)_4, && (\mathrm{d}^*_\omega\psi_+)_4 = -(\mathrm{d}^*\psi_-)_4,\\
&(\mathrm{d}^*\psi_+)_5 = -(\mathrm{d}^*_\omega\psi_-)_5 = n\eta\lrcorner\psi_+, && (\mathrm{d}^*_\omega\psi_+)_5 = -(\mathrm{d}^*\psi_-)_5 = n\eta\lrcorner\psi_-,\\
&i_I(\mathrm{d}^*\psi_\pm)_{1,2} = (n-3)(\mathrm{d}^*_\omega\psi_\pm)_{1,2}, && i_I(\mathrm{d}^*_\omega\psi_\pm)_{1,2} = -(n-3)(\mathrm{d}^*\psi_\pm)_{1,2},\\
&i_I(\mathrm{d}^*\psi_\pm)_4 = -(n-1)(\mathrm{d}^*_\omega\psi_\pm)_4, && i_I(\mathrm{d}^*_\omega\psi_\pm)_4 = (n-1)(\mathrm{d}^*\psi_\pm)_4.
\end{aligned}$$

Proof. Here we only consider $n \geq 4$, the proof for $n = 3$ will be shown in next section. The identities of fourth and fifth lines follow by similar arguments to those contained in the proof of Lemma 2.8. The identities of the third line follow by a straightforwardly way.

Making use of the maps Ξ_+, Ξ_- and Eq. (2.6), we note

$$(\nabla\cdot\psi_-)_{1,2} = (\nabla_I\cdot\psi_+)_{1,2}, \qquad (\nabla\cdot\psi_-)_{3,4} = -(\nabla_I\cdot\psi_+)_{3,4}. \tag{2.11}$$

Hence, applying the maps d^* and d^*_ω to both sides of these equalities, the identities of first and second lines in Lemma follow. □

We know how to compute $(\nabla\psi_+)_4$ and $(\nabla\psi_+)_5$ (Eq. (2.8) and Theorem 2.6) . Now, we will show expressions for the remaining $SU(n)$-parts of $\nabla\psi_+$ in terms of $\mathrm{d}\omega$ and $\mathrm{d}\psi_+$.

Proposition 2.10. *Let M be a special almost Hermitian 2n-manifold, $n \geq 4$. Then*

(i) $(\nabla\cdot\psi_+)_1 = e_i \otimes Ie_i \wedge b \wedge \omega + e_i \otimes e_i \wedge I_{(1)}b \wedge \omega$, $(\mathrm{d}\psi_+)_1 = -2b \wedge \omega \wedge \omega$, $(\mathrm{d}^*\psi_+)_1 = 2I_{(1)}b \wedge \omega$ *and* $(\mathrm{d}^*_\omega\psi_+)_1 = -2b \wedge \omega$, *where* b *is given by* $b = Im(*_{\mathbb{C}}((\nabla\omega)_1 + iI_{(1)}(\nabla\omega)_1))$, $12(-1)^n b = I * (\mathrm{d}\psi_+ \wedge \omega)$, $12(-1)^n I_{(1)}b = I * (\mathrm{d}^*\psi_+ \wedge \omega \wedge \omega)$, or $12(-1)^{n-1}b = I * (\mathrm{d}^*_\omega\psi_+ \wedge \omega \wedge \omega)$;

(ii) $(\nabla\cdot\psi_+)_2 = e_i \otimes e_i \lrcorner(\mathrm{d}^*_\omega\psi_+)_{1,2} \wedge \omega + e_i \otimes Ie_i \lrcorner(\mathrm{d}^*\psi_+)_{1,2} \wedge \omega - 8(\nabla\psi_+)_1$, *where* $4(n-2)(a)_{1,2} = (n-1)(n-2)a + 2L(a)$, *for* $a = \mathrm{d}^*\psi_+, \mathrm{d}^*_\omega\psi_+$;

(iii) $2(\nabla\cdot\psi_+)_3 = \Xi_+((1 - I_{(2)}I_{(3)})(\mathrm{d}\omega)_3)$, *where* $(\mathrm{d}\omega)_3 = (\mathrm{d}\omega)_{3,4} - (\mathrm{d}\omega)_4$ *with* $4(\mathrm{d}\omega)_{3,4} = 3\mathrm{d}\omega + L(\mathrm{d}\omega)$ *and* $(n-1)(\mathrm{d}\omega)_4 = -I\mathrm{d}^*\omega \wedge \omega$.

Proof. By Theorem 2.4, $(\nabla \cdot \psi_+)_1 = e_i \otimes Ie_i \wedge b \wedge \omega + e_i \otimes e_i \wedge I_{(1)}b \wedge \omega$, where $b \in [\![\lambda^{n-3,0}]\!]$. Therefore, $(\mathrm{d}\psi_+)_1 = -2b \wedge \omega \wedge \omega$. On the other hand, it is not hard to check

$$* (c \wedge \omega \wedge \omega \wedge \omega) = 6(-1)^{n-1} Ic, \tag{2.12}$$

for all $c \in [\![\lambda^{n-3,0}]\!]$. Therefore, taking this last identity into account, we have

$$*(\mathrm{d}\psi_+ \wedge \omega) = *((\mathrm{d}\psi_+)_1 \wedge \omega) = 12(-1)^{n-1} Ib.$$

Now, let us assume $\nabla\omega = c \in [\![\lambda^{3,0}]\!] = \mathcal{W}_1$. Then, computing $\Xi_+(c)$, we have

$$\Xi_+(c) = e_i \otimes Ie_i \wedge \mathrm{Im}(*_{\mathbb{C}}(c + \mathrm{i}I_{(1)}c)) \wedge \omega - e_i \otimes e_i \wedge \mathrm{Re}(*_{\mathbb{C}}(c + \mathrm{i}I_{(1)}c)) \wedge \omega.$$

Hence the first identity for b follows. The remaining identities of (i) involving $\mathrm{d}^*\psi_+$ and $\mathrm{d}^*_\omega\psi_+$ follow by a straightforward way from $(\nabla\psi_+)_1$, taking Eq. (2.12) into account.

For part (ii). If $\nabla\psi_+ = x \otimes b \wedge \omega \in T^*M \otimes [\![\lambda^{n-2,0}]\!]$, by Eq. (2.10), we have $2(\nabla\psi_+)_{1,2} = (x \otimes b \wedge \omega + Ix \otimes I_{(1)}b \wedge \omega)$. Therefore, making use of part (i), it follows

$$6(\nabla\psi_+)_1 = e_i \otimes Ie_i \wedge (x \lrcorner I_{(1)}b) \wedge \omega + e_i \otimes e_i \wedge I_{(1)}(x \lrcorner I_{(1)}b) \wedge \omega.$$

Moreover,

$$2(\mathrm{d}^*\psi_+)_{1,2} = Ix \wedge b - x \wedge I_{(1)}b - 2(x \lrcorner b) \wedge \omega, \tag{2.13}$$

$$2(\mathrm{d}^*_\omega\psi_+)_{1,2} = x \wedge b + Ix \wedge I_{(1)}b - 2(x \lrcorner b) \wedge \omega. \tag{2.14}$$

From these equations, it is not hard to check

$$e_i \otimes e_i \lrcorner (\mathrm{d}^*_\omega\psi_+)_{1,2} \wedge \omega + e_i \otimes Ie_i \lrcorner (\mathrm{d}^*\psi_+)_{1,2} \wedge \omega = 2(\nabla\psi_+)_{1,2} + 6(\nabla\psi_+)_1.$$

Hence the first identity of (ii) follows. Furthermore, by Lemma 2.8, we have the equalities

$$-2L(\mathrm{d}^*\psi_+)_{1,2} = (n-2)(n-5)(\mathrm{d}^*\psi_+)_{1,2},$$
$$-2L(\mathrm{d}^*\psi_+)_{4,5} = (n-1)(n-2)(\mathrm{d}^*\psi_+)_{4,5}.$$

Therefore,

$$4(n-2)(\mathrm{d}^*\psi_+)_{1,2} = (n-1)(n-2)\mathrm{d}^*\psi_+ + 2L(\mathrm{d}^*\psi_+),$$

$$4(n-2)(\mathrm{d}^*\psi_+)_{4,5} = -(n-2)(n-5)\mathrm{d}^*\psi_+ - 2L(\mathrm{d}^*\psi_+).$$

The required expression for $(\mathrm{d}^*_\omega\psi_+)_{1,2}$ can be deduced in a similar way.

Finally, part (iii) follows from identities for $\nabla\omega$ given in [6,8]. □

Remark 2.11.

(i) From the identities given in Lemma 2.9, the forms $\mathrm{d}^*\psi_+$ and $\mathrm{d}^*_\omega\psi_+$ can be computed in terms of $\mathrm{d}\psi_+$ $(\mathrm{d}\psi_-)$. Thus Proposition 2.10 corroborates our claiming that, for $n \geq 4$, $\mathrm{d}\omega$ and $\mathrm{d}\psi_+$ $(\mathrm{d}\psi_-)$ are enough to know the intrinsic $SU(n)$-torsion.

(ii) Taking Eq. (2.11) into account, it is not hard to deduce the respective $SU(n)$-components, $n \geq 4$, of $\nabla\psi_-$ from those of $\nabla\psi_+$.

Relative with conformal changes of metric, we point out the following facts which are generalizations of results for $SU(3)$-structures proved by Chiossi and Salamon [3].

Proposition 2.12. *For conformal changes of metric given by* $\langle\cdot,\cdot\rangle_o = e^{2f}\langle\cdot,\cdot\rangle$, *the* $\mathcal{W}_4$ *and* $\mathcal{W}_5$ *parts of the intrinsic* $SU(n)$*-torsion,* $n \geq 2$, *are modified in the way*

$$I\mathrm{d}^*\omega_o = I\mathrm{d}^*\omega - 2(n-1)\mathrm{d}f, \qquad \eta_o = \eta - \frac{1}{n}\mathrm{d}f,$$

where ω_o *and* η_o *are respectively the Kähler form and the* $\mathcal{W}_5$ *one-form of the metric* $\langle\cdot,\cdot\rangle_o$. *Moreover, the one-form* $2n(n-1)\eta - I\mathrm{d}^*\omega$ *is not altered by such changes of metric.*

Proof. On one hand, the equation for $I\mathrm{d}^*\omega_o$ was deduced in [8]. On the other hand, from $\psi_{+o} = e^{nf}\psi_+$ and $\psi_{-o} = e^{nf}\psi_-$, we have $\mathrm{d}\psi_{+o} = ne^{nf}\mathrm{d}f \wedge \psi_+ + e^{nf}\mathrm{d}\psi_+$ and $\mathrm{d}\psi_{-o} = ne^{nf}\mathrm{d}f \wedge \psi_- + e^{nf}\mathrm{d}\psi_-$. Moreover, if $*_o$ is the Hodge star operator for $\langle\cdot,\cdot\rangle_o$ and α is a p-form, then $*_o\alpha = e^{2(n-p)f} * \alpha$. Taking this last identity into account, we deduce

$$\begin{aligned} &*_o(*_o\mathrm{d}\psi_{+o} \wedge \psi_{+o}) + *_o(*_o\mathrm{d}\psi_{-o} \wedge \psi_{-o}) \\ &\quad = *(*\mathrm{d}\psi_+ \wedge \psi_+) + *(*\mathrm{d}\psi_- \wedge \psi_-) - n2^{n-1}\mathrm{d}f. \end{aligned}$$

The required identity for η_o follows from this last identity and Theorem 2.6. Finally, it is obvious that $2n(n-1)\eta_o - I\mathrm{d}^*\omega_o = 2n(n-1)\eta - I\mathrm{d}^*\omega$. □

Remark 2.13. By Proposition 2.12, for $n = 3$, the one-form $12\eta - I\mathrm{d}^*\omega$ is not altered by conformal changes of metric. In [3], Chiossi and Salamon consider six-dimensional manifolds with $SU(3)$-structure and prove that the tensor $3\tau_{\mathcal{W}_4} + 2\tau_{\mathcal{W}_5}$ is not modified under conformal changes of metric, where $\tau_{\mathcal{W}_4}$ and $\tau_{\mathcal{W}_5}$ are one-forms such that, in the terminology here used, are given by $2\tau_{\mathcal{W}_4} = -I\mathrm{d}^*\omega$ and $2\tau_{\mathcal{W}_5} = \eta + I\mathrm{d}^*\omega$. Note that $3\tau_{\mathcal{W}_4} + 2\tau_{\mathcal{W}_5} = \frac{1}{2}(12\eta - I\mathrm{d}^*\omega)$.

3. Low dimensions

In this section we consider special almost Hermitian manifolds of dimension two, four and six.

3.1. Six dimensions

Here we focus our attention on the very special case of six-dimensional manifolds with $SU(3)$-structure (see [3]). In this case, we have

$$\nabla\omega \in T^*M \otimes \mathfrak{u}(3)^\perp = \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_4. \tag{3.1}$$

If we denote $[T^*M_{\mathbb{C}} \otimes_{\mathbb{C}} \Lambda^2 T^*M_{\mathbb{C}}] = \{Re(b_{\mathbb{C}}) \,|\, b_{\mathbb{C}} \in T^*M_{\mathbb{C}} \otimes_{\mathbb{C}} \Lambda^2 T^*M_{\mathbb{C}}\}$, some summands in (3.1) are described by

$$\begin{aligned}
&\mathcal{W}_1^+ = \mathbb{R}\psi_+, \qquad \mathcal{W}_1^- = \mathbb{R}\psi_-,\\
&\mathcal{W}_1^+ + \mathcal{W}_2^+ = \{b \in [T^*M_{\mathbb{C}} \otimes_{\mathbb{C}} \Lambda^2 T^*M_{\mathbb{C}}] \,|\, \langle \cdot \lrcorner \psi_+, \cdot \lrcorner b\rangle \text{ is symmetric}\},\\
&\mathcal{W}_1^- + \mathcal{W}_2^- = \{b \in [T^*M_{\mathbb{C}} \otimes_{\mathbb{C}} \Lambda^2 T^*M_{\mathbb{C}}] \,|\, \langle \cdot \lrcorner \psi_+, \cdot \lrcorner b\rangle \text{ is skew-symmetric}\}.
\end{aligned}$$

By Proposition 2.3, the $SU(3)$-maps Ξ_+ and Ξ_- are injective and

$$\Xi_+(T^*M \otimes \mathfrak{u}(3)^\perp) = \Xi_-(T^*M \otimes \mathfrak{u}(3)^\perp) = T^*M \otimes T^*M \wedge \omega.$$

In the following theorem we describe properties of the $SU(3)$-components of $\nabla\psi_+$ and $\nabla\psi_-$.

Theorem 3.1. *Let M be a special almost Hermitian six-manifold with Kähler form ω and complex volume form $\Psi = \psi_+ + \mathrm{i}\psi_-$. Then*

$$\begin{aligned}
\nabla\psi_+ &\in \mathcal{W}_1^{\Xi;a} + \mathcal{W}_1^{\Xi;b} + \mathcal{W}_2^{\Xi;a} + \mathcal{W}_2^{\Xi;b} + \mathcal{W}_3^{\Xi} + \mathcal{W}_4^{\Xi} + \mathcal{W}_5^-,\\
\nabla\psi_- &\in \mathcal{W}_1^{\Xi;a} + \mathcal{W}_1^{\Xi;b} + \mathcal{W}_2^{\Xi;a} + \mathcal{W}_2^{\Xi;b} + \mathcal{W}_3^{\Xi} + \mathcal{W}_4^{\Xi} + \mathcal{W}_5^+,
\end{aligned}$$

where $\mathcal{W}_i^{\Xi;a} = \Xi_+(\mathcal{W}_i^+) = \Xi_-(\mathcal{W}_i^-)$, $\mathcal{W}_i^{\Xi;b} = \Xi_+(\mathcal{W}_i^-) = \Xi_-(\mathcal{W}_i^+)$, $i = 1, 2$; $\mathcal{W}_j^{\Xi} = \Xi_+(\mathcal{W}_j) = \Xi_-(\mathcal{W}_j)$, $j = 3, 4$; $\mathcal{W}_5^+ = T^*M \otimes \psi_+$ and $\mathcal{W}_5^- = T^*M \otimes \psi_-$. If $\mathcal{W}^{\Xi} = T^*M \otimes T^*M \wedge \omega$, $\mathcal{L}$ is the map defined by (2.7) and $\tilde{a}$ denotes the alternation map, the modules $\mathcal{W}_i^{\Xi;a}$, $\mathcal{W}_i^{\Xi;b}$ and $\mathcal{W}_j^{\Xi}$ are described by

$$\begin{aligned}
&\mathcal{W}_1^{\Xi;a} = \mathbb{R}e_i \otimes e_i \wedge \omega, \qquad \mathcal{W}_1^{\Xi;b} = \mathbb{R}e_i \otimes Ie_i \wedge \omega,\\
&\mathcal{W}_2^{\Xi;a} = \{b \in \mathcal{W}^{\Xi} \,|\, \langle b(e_i, e_i, \cdot, \cdot), \omega\rangle = 0,\ b(e_i, Ie_i, \cdot, \cdot) = 0 \text{ and } \mathcal{L}(b) = b\},\\
&\mathcal{W}_2^{\Xi;b} = \{b \in \mathcal{W}^{\Xi} \,|\, \langle b(e_i, Ie_i, \cdot, \cdot), \omega\rangle = 0,\ b(e_i, e_i, \cdot, \cdot) = 0 \text{ and } \mathcal{L}(b) = b\},\\
&\mathcal{W}_1^{\Xi;a} + \mathcal{W}_1^{\Xi;b} + \mathcal{W}_2^{\Xi;a} + \mathcal{W}_2^{\Xi;b} = \{b \in \mathcal{W}^{\Xi} \,|\, \mathcal{L}(b) = b\},\\
&\mathcal{W}_3^{\Xi} = \{b \in \mathcal{W}^{\Xi} \,|\, \mathcal{L}(b) = -b \text{ and } \tilde{a}(b) = 0\},\\
&\mathcal{W}_4^{\Xi} = \{e_i \otimes ((x \wedge e_i) \lrcorner \psi_+) \wedge \omega \,|\, x \in TM\} = \{e_i \otimes ((x \wedge e_i) \lrcorner \psi_-) \wedge \omega \,|\, x \in TM\},\\
&\mathcal{W}_3^{\Xi} + \mathcal{W}_4^{\Xi} = \{b \in \mathcal{W}^{\Xi} \,|\, \mathcal{L}(b) = -b\}.
\end{aligned}$$

Proof. We can proceed in a similar way as in the proof of Theorem 2.4. □

If we consider the alternation maps $\tilde{a}_\pm : T^*M \otimes T^*M \wedge \omega + \mathcal{W}_5^\mp \to \Lambda^4 T^*M$, we get the following consequences of Theorem 3.1.

Corollary 3.2. *For $SU(3)$-structures, the exterior derivatives of ψ_+ and ψ_- are such that*

$$\mathrm{d}\psi_+, \mathrm{d}\psi_- \in \Lambda^4 T^*M = \mathcal{W}_1^a + \mathcal{W}_2^a + \mathcal{W}_{4,5}^a,$$

where $\tilde{a}_\pm(\mathcal{W}_1^{\Xi;b}) = \mathcal{W}_1^a$, $\tilde{a}_\pm(\mathcal{W}_2^{\Xi;b}) = \mathcal{W}_2^a$ and $\tilde{a}_\pm(\mathcal{W}_4^{\Xi}) = \tilde{a}_\pm(\mathcal{W}_5^\mp) = \mathcal{W}_{4,5}^a$. Moreover, $Ker(\tilde{a}_\pm) = \mathcal{W}_1^{\Xi;a} + \mathcal{W}_2^{\Xi;a} + \mathcal{W}_3^{\Xi} + \mathcal{A}_\pm$, where $T^*M \cong \mathcal{A}_\pm \subseteq \mathcal{W}_4^{\Xi} + \mathcal{W}_5^\mp$, and the mod-

ules $\mathcal{W}_i^a$ are described by

$$\begin{aligned}
&\mathcal{W}_1^a = \mathbb{R}\omega \wedge \omega,\\
&\mathcal{W}_2^a = \mathfrak{su}(3) \wedge \omega = \{b \in \Lambda^4 T^*M \mid b \wedge \omega = 0 \, and \, *b \wedge \psi_+ = 0\}\\
&\quad = \{b \in \Lambda^4 T^*M \mid b \wedge \omega = 0 \, and \, *b \wedge \psi_- = 0\},\\
&\mathcal{W}_{4,5}^a = T^*M \wedge \psi_+ = T^*M \wedge \psi_- = [\![\lambda^{2,0}]\!] \wedge \omega\\
&\quad = \{x \lrcorner \psi_+ \wedge \omega \mid x \in TM\} = \{x \lrcorner \psi_- \wedge \omega \mid x \in TM\}.
\end{aligned}$$

Moreover, we also have

$$\begin{aligned}
&\mathcal{W}_1^a + \mathcal{W}_2^a = \{b \in \Lambda^4 T^*M \mid *b \wedge \psi_+ = 0\} = \{b \in \Lambda^4 T^*M \mid *b \wedge \psi_- = 0\},\\
&\mathcal{W}_2^a + \mathcal{W}_{4,5}^a = \{b \in \Lambda^4 T^*M \mid b \wedge \omega = 0\}.
\end{aligned}$$

In this point, one can proceed as in the proof, for high dimensions, of Theorem 2.6 and obtain the results of such Theorem for $n = 3$. Along such a proof we would get

$$(\nabla\psi_+)_4 = \Xi_+(\nabla\omega)_4 = -\tfrac{1}{4} e_i \otimes ((\mathrm{d}^*\omega \wedge e_i) \lrcorner \psi_+) \wedge \omega, \tag{3.2}$$

$$(\mathrm{d}\psi_+)_{4,5} = -(3\eta + \tfrac{1}{2} I \mathrm{d}^*\omega) \wedge \psi_+. \tag{3.3}$$

Likewise, in a similar way, we would also obtain

$$(\nabla\psi_-)_4 = \Xi_-(\nabla\omega)_4 = -\tfrac{1}{4} e_i \otimes ((\mathrm{d}^*\omega \wedge e_i) \lrcorner \psi_-) \wedge \omega, \tag{3.4}$$

$$(\mathrm{d}\psi_-)_{4,5} = -(3\eta + \tfrac{1}{2} I \mathrm{d}^*\omega) \wedge \psi_-. \tag{3.5}$$

Remark 3.3.

(i) From Eq. (3.3), it follows that $\mathcal{A}_+ \subseteq ker(\tilde{a}_+)$ is given by

$$\mathcal{A}_+ = \{-\tfrac{1}{4} e_i \otimes ((x \wedge e_i) \lrcorner \psi_+) \wedge \omega - \tfrac{1}{2} x \otimes \psi_- \mid x \in TM\}.$$

Analogously, from Eq. (3.3), for $\mathcal{A}_- \subseteq ker(\tilde{a}_-)$, we have

$$\mathcal{A}_- = \{-\tfrac{1}{4} e_i \otimes ((x \wedge e_i) \lrcorner \psi_-) \wedge \omega + \tfrac{1}{2} x \otimes \psi_+ \mid x \in TM\}.$$

(ii) Theorem 2.6 says that η can be computed from $\mathrm{d}\omega$ and $\mathrm{d}\psi_+$ ($\mathrm{d}\psi_-$). Moreover, since $\mathrm{d}\omega \in \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_3 + \mathcal{W}_4$ and

$$\begin{aligned}
&\mathrm{d}\psi_+ \in \mathcal{W}_1^a + \mathcal{W}_2^a + \mathcal{W}_{4,5}^a = \tilde{a}_+ \circ \Xi_+(\mathcal{W}_1^- + \mathcal{W}_2^- + \mathcal{W}_4) + \tilde{a}_+(\mathcal{W}_5^-),\\
&\mathrm{d}\psi_- \in \mathcal{W}_1^a + \mathcal{W}_2^a + \mathcal{W}_{4,5}^a = \tilde{a}_- \circ \Xi_-(\mathcal{W}_1^+ + \mathcal{W}_2^+ + \mathcal{W}_4) + \tilde{a}_-(\mathcal{W}_5^+),
\end{aligned}$$

we need $\mathrm{d}\omega$, $\mathrm{d}\psi_+$ and $\mathrm{d}\psi_-$ to have the whole information about the intrinsic $SU(3)$-torsion.

The $\mathcal{W}_4$ and $\mathcal{W}_5$ parts of $\nabla\psi_+$ are given by Eq. (3.2) and Theorem 2.6. As in the previous section, for sake of completeness, we will see how to compute the remaining parts of $\nabla\psi_+$ by using $\mathrm{d}\omega$, $\mathrm{d}\psi_+$ and $\mathrm{d}\psi_-$. For such a purpose, we study properties of the coderivatives $\mathrm{d}^*\psi_+$, $\mathrm{d}^*\psi_-$ and the two-forms $\mathrm{d}^*_\omega\psi_+$ and $\mathrm{d}^*_\omega\psi_-$. Note that, by Lemma 2.1, we have $\mathrm{d}^*\psi_+ = *\mathrm{d}\psi_-$ and $\mathrm{d}^*\psi_- = - * \mathrm{d}\psi_+$. Therefore,

$$\mathrm{d}^*\psi_+, \mathrm{d}^*\psi_-, \mathrm{d}^*_\omega\psi_+, \mathrm{d}^*_\omega\psi_- \in \Lambda^2 T^*M = \mathcal{W}_1^c + \mathcal{W}_2^c + \mathcal{W}_{4,5}^c,$$

where $\mathcal{W}_1^c = *(\mathcal{W}_1^a)$, $\mathcal{W}_2^c = *(\mathcal{W}_2^a)$ and $\mathcal{W}_{4,5}^c = *(\mathcal{W}_{4,5}^a)$.

Lemma 3.4. *For $SU(3)$-structures, the modules $\mathcal{W}_1^c$, $\mathcal{W}_2^c$ and $\mathcal{W}_{4,5}^c$ are defined by*:

$$\begin{aligned}
&\mathcal{W}_1^c = \mathbb{R}\omega, \quad \mathcal{W}_2^c = \{b \in \Lambda^2T^*M \mid b\wedge\omega\wedge\omega = 0 \, and \, b \wedge \psi_+ = 0\},\\
&\mathcal{W}_1^c + \mathcal{W}_2^c = \{b \in \Lambda^2T^*M \mid Ib = b\} = \{b \in \Lambda^2T^*M \mid b\wedge\psi_+ = 0\},\\
&\mathcal{W}_{4,5}^c = [\![\lambda^{2,0}]\!] = \{x \lrcorner \psi_+ \mid x \in TM\},\\
&\mathcal{W}_2^c + \mathcal{W}_{4,5}^c = \{b \in \Lambda^2T^*M \mid b\wedge\omega\wedge\omega = 0\}.
\end{aligned}$$

Proof. It follows by similar arguments as in the proof of Lemma 2.8. □

Now one can prove the identities given in Lemma 2.9 for $n = 3$. Such a proof can be constructed in a similar way that the one for $n \geq 4$, taking analog results for $SU(3)$-structures into account. Such identities will be used in the following proposition, where we compute some $SU(3)$-parts of $\nabla\psi_+$.

Proposition 3.5. *Let M be a special almost Hermitian six-manifold. Then*

(i) $(\nabla\cdot\psi_+)_{1;a} = -w_1^+ e_i \otimes e_i \wedge \omega$, $(\mathrm{d}\psi_-)_1 = 2w_1^+\,\omega\wedge\omega$ *and* $(\mathrm{d}^*\psi_+)_1 = 4w_1^+\omega$, *where* w_1^+ *is given by* $12w_1^+ = *(\mathrm{d}\psi_- \wedge \omega) = \langle *\mathrm{d}\psi_-, \omega\rangle$ *or* $(\nabla\omega)_{1;+} = w_1^+\,\psi_+$;

(ii) $(\nabla\cdot\psi_+)_{1;b} = w_1^- e_i \otimes Ie_i \wedge \omega$, $(\mathrm{d}\psi_+)_1 = -2w_1^-\,\omega\wedge\omega$ *and* $(\mathrm{d}^*\psi_-)_1 = 4w_1^-\omega$, *where* w_1^- *is given by* $-12w_1^- = *(\mathrm{d}\psi_+\wedge\omega) = \langle *\mathrm{d}\psi_+, \omega\rangle$ *or* $(\nabla\omega)_{1;-} = w_1^-\,\psi_-$;

(iii) $4(\nabla\cdot\psi_+)_{1,2;a} = -\langle *d\psi_-, \omega\rangle\, e_i\otimes e_i\wedge\omega + \iota_\omega(I_{(2)} - I_{(1)}) * \mathrm{d}\psi_-$, *where* $\iota_\omega : T^*M \otimes T^*M \to T^*M \otimes T^*M \wedge \omega$ *defined by* $\iota_\omega(a\otimes b) = a\otimes b\wedge\omega$;

(iv) $-4(\nabla\cdot\psi_+)_{1,2;b} = \langle *\mathrm{d}\psi_+, \omega\rangle\, e_i\otimes Ie_i\wedge\omega + \iota_\omega(1 + I_{(1)}I_{(2)}) * \mathrm{d}\psi_+$ *and* $-2(\mathrm{d}\psi_+)_{1,2} = -\langle *\mathrm{d}\psi_+, \omega\rangle\,\omega\wedge\omega + \omega\wedge(1 + I_{(1)}I_{(2)}) * \mathrm{d}\psi_+$;

(v) $2(\nabla\cdot\psi_+)_3 = \Xi_+((1 - I_{(2)}I_{(3)})(\mathrm{d}\omega)_3)$, *where* $(\mathrm{d}\omega)_3 = (\mathrm{d}\omega)_{3,4} - (\mathrm{d}\omega)_4$ *with* $4(\mathrm{d}\omega)_{3,4} = 3\mathrm{d}\omega + L(\mathrm{d}\omega)$ *and* $2(\mathrm{d}\omega)_4 = -I\mathrm{d}^*\omega\wedge\omega$.

Proof. For part (i). If $(\nabla\omega)_{1;+} = w_1^+\psi_+$, $w_1^+ \in \mathbb{R}$, by Theorem 3.1, we obtain $(\nabla\psi_+)_{1;a} = \Xi_+(\nabla\omega)_{1;+} = -w_1^+ e_i\otimes e_i\wedge\omega$ and $(\nabla\psi_-)_{1;b} = \Xi_-(\nabla\omega)_{1;+} = -w_1^+ e_i\otimes Ie_i\wedge\omega$. Therefore, $(\mathrm{d}\psi_-)_1 = 2w_1^+\omega\wedge\omega$. On the other hand, since $\omega\wedge\omega\wedge\omega = 6\,Vol$, we have $*(\mathrm{d}\psi_-\wedge\omega) = *((\mathrm{d}\psi_-)_1\wedge\omega) = 12w_1^+ = \langle *\mathrm{d}\psi_-, \omega\rangle$.

For part (ii). By an analog way, since $(\nabla\omega)_{1;-} = w_1^-\,\psi_-$, $w_1^- \in \mathbb{R}$, we have $(\nabla.\psi_+)_{1;b} = \Xi_+(\nabla\omega)_{1;-} = w_1^- e_i\otimes Ie_i\wedge\omega$. Therefore, $(\mathrm{d}\psi_+)_1 = -2w_1^-\omega\wedge\omega$. Hence we have $*(\mathrm{d}\psi_+\wedge\omega) = *((\mathrm{d}\psi_+)_1\wedge\omega) = -12w_1^- = \langle *\mathrm{d}\psi_+, \omega\rangle$.

For part (iii). If $\nabla\psi_+ = x\otimes y\wedge\omega$, by Theorem 3.1, we have

$$4(\nabla\psi_+)_{1,2;a} = x\otimes y\wedge\omega + y\otimes x\wedge\omega + Ix\otimes Iy\wedge\omega + Iy\otimes Ix\wedge\omega.$$

Therefore, $2(\mathrm{d}^*\psi_+)_{1,2} = -2\langle x,y\rangle\omega + Ix\wedge y - x\wedge Iy$. Since $\langle \mathrm{d}^*\psi_+,\omega\rangle = 12w_1^+ = -2\langle x,y\rangle$ and $2I_{(1)}(\mathrm{d}^*\psi_+)_{1,2} = 2\langle x,y\rangle\langle\cdot,\cdot\rangle - (x\otimes y + y\otimes x + Ix\otimes Iy + Iy\otimes Ix)$, we have

$$2\iota_\omega I_{(1)}(\mathrm{d}^*\psi_+)_{1,2} + \langle \mathrm{d}^*\psi_+,\omega\rangle e_i\otimes e_i\wedge\omega = -4(\nabla\psi_+)_{1,2;a}.$$

On the other hand, since $I(\mathrm{d}^*\psi_+)_{1,2} = -(\mathrm{d}^*\psi_+)_{1,2}$ and $I(\mathrm{d}^*\psi_+)_{4,5} = -(\mathrm{d}^*\psi_+)_{4,5}$, it follows $2(\mathrm{d}^*\psi_+)_{1,2} = \mathrm{d}^*\psi_+ + I\mathrm{d}^*\psi_+$. Thus,

$$\iota_\omega(I_{(1)} - I_{(2)})\mathrm{d}^*\psi_+ + \langle \mathrm{d}^*\psi_+,\omega\rangle e_i\otimes e_i\wedge\omega = -4(\nabla\psi_+)_{1,2;a}.$$

Finally, taking $\mathrm{d}^*\psi_+ = *\mathrm{d}\psi_-$ into account, the required identity in (iii) follows.

For part (iv). We proceed in a similar way as in the proof for (iii), but now we consider

$$4(\nabla\psi_+)_{1,2;b} = x\otimes y\wedge\omega - y\otimes x\wedge\omega + Ix\otimes Iy\wedge\omega - Iy\otimes Ix\wedge\omega$$

and we compute $(\mathrm{d}^*_\omega\psi_+)_{1,2}$. Thus we have $2(\mathrm{d}^*_\omega\psi_+)_{1,2} = -2\omega(x,y)\omega + x\wedge y + Ix\wedge Iy$. Since $(\mathrm{d}^*_\omega\psi_+)_1 = (\mathrm{d}^*\psi_-)_1 = 4w_1^-\omega = -\frac{2}{3}\omega(x,y)\omega$, we obtain

$$2\iota_\omega(\mathrm{d}^*_\omega\psi_+)_{1,2} + \langle \mathrm{d}^*_\omega\psi_+,\omega\rangle e_i\otimes Ie_i\wedge\omega = 4(\nabla\psi_+)_{1,2;b}.$$

Finally, taking $2(\mathrm{d}^*_\omega\psi_+)_{1,2} = 2(\mathrm{d}^*\psi_-)_{1,2} = \mathrm{d}^*\psi_- + I\mathrm{d}^*\psi_-$ and $\mathrm{d}^*\psi_- = -*d\psi_+$ into account, it follows the first required identity in (iv). By alternating both sides of such an identity, the second required equation follows. Part (v) follows as in the proof of Proposition 2.10 for $(\nabla\psi_+)_3$. □

Remark 3.6. From the maps Ξ_+, Ξ_- and identities (2.6), it is not hard to prove

$$(\nabla\cdot\psi_-)_{1,2;a} = (\nabla_I\cdot\psi_+)_{1,2;b}, \qquad (\nabla\cdot\psi_-)_{1,2;b} = -(\nabla_I\cdot\psi_+)_{1,2;a},$$
$$(\nabla\cdot\psi_-)_{3,4} = -(\nabla_I\cdot\psi_+)_{3,4}.$$

Thus, taking these identities into account, one can deduce the respective $SU(3)$-components of $\nabla\psi_-$ from those of $\nabla\psi_+$.

The following results are relative to nearly Kähler six-manifolds.

Theorem 3.7. *Let M be a special almost Hermitian connected six-manifold of type $\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_5$ which is not of type $\mathcal{W}_5$ such that $\nabla\omega = w_1^+\psi_+ + w_1^-\psi_-$, then*

(i) *$\nabla\omega$ is nowhere zero, $\alpha = (w_1^+)^2 + (w_1^-)^2$ is a positive constant and* $\mathrm{d}w_1^+ = -w_1^- I\eta$, $\mathrm{d}w_1^- = w_1^+ I\eta$;
(ii) *the one-form $I\eta$ is closed and given by $3\alpha I\eta = w_1^+\mathrm{d}w_1^- - w_1^-\mathrm{d}w_1^+$;*
(iii) *M is of type $\mathcal{W}_1^+ + \mathcal{W}_1^-$ if and only if w_1^+ and w_1^- are constant.*
(iv) *If M is of type $\mathcal{W}_1^+ + \mathcal{W}_5$, then M is of type $\mathcal{W}_1^+$ or of type $\mathcal{W}_5$.*
(v) *If M is of type $\mathcal{W}_1^- + \mathcal{W}_5$, then M is of type $\mathcal{W}_1^-$ or of type $\mathcal{W}_5$.*

Proof. Since *M* is of dimension six, it is straightforward to check

$$(x\lrcorner\psi_+)\wedge\psi_+ = (x\lrcorner\psi_-)\wedge\psi_- = x\wedge\omega\wedge\omega = -2*Ix, \tag{3.6}$$

$$(x\lrcorner\psi_+)\wedge\psi_- = -(x\lrcorner\psi_-)\wedge\psi_+ = Ix\wedge\omega\wedge\omega = 2*x, \tag{3.7}$$

for all vector *x*.

Since *M* is of type $\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_5$, we have

$$\mathrm{d}\omega = 3w_1^+\psi_+ + 3w_1^-\psi_-, \tag{3.8}$$

$$\mathrm{d}\psi_+ = -2w_1^-\omega\wedge\omega - 3I\eta\wedge\psi_-, \tag{3.9}$$

$$\mathrm{d}\psi_- = 2w_1^+\omega\wedge\omega + 3I\eta\wedge\psi_+. \tag{3.10}$$

Now, differentiating Eqs. (3.9) and (3.10) and using Eq. (3.8), we have

$$0 = 2(\mathrm{d}w_1^- - 3w_1^+ I\eta)\wedge\omega\wedge\omega + 3\mathrm{d}I\eta\wedge\psi_-, \tag{3.11}$$

$$0 = 2(\mathrm{d}w_1^+ + 6w_1^- I\eta)\wedge\omega\wedge\omega + 3\mathrm{d}I\eta\wedge\psi_+. \tag{3.12}$$

But $\mathrm{d}I\eta \in \Lambda^2T^*M = \mathbb{R}\omega + \mathfrak{su}(3) + \mathfrak{u}(3)^\perp$ and $\mathrm{d}I\eta_{\,\mathfrak{u}(3)^\perp} = x\lrcorner\psi_+$. Therefore,

$$\mathrm{d}I\eta\wedge\psi_+ = (x\lrcorner\psi_+)\wedge\psi_+, \qquad \mathrm{d}I\eta\wedge\psi_- = (x\lrcorner\psi_+)\wedge\psi_-.$$

Taking these identities into account and making use of Eqs. (3.6) and (3.7), from Eqs. (3.11) and (3.12) it follows

$$\tfrac{3}{2}x = I\mathrm{d}w_1^- + 3w_1^+\eta = -\mathrm{d}w_1^+ - w_1^- I\eta. \tag{3.13}$$

On the other hand, differentiating Eq. (3.8), making use of Eqs. (3.9) and (3.10), and taking $x\wedge\psi_+ = Ix\wedge\psi_-$ into account, we obtain

$$0 = (\mathrm{d}w_1^+ + 3w_1^- I\eta - I\mathrm{d}w_1^- - 3w_1^+\eta)\wedge\psi_+.$$

Therefore, taking Eq. (3.13) into account, we get $I\mathrm{d}w_1^- + 3w_1^+\eta = \mathrm{d}w_1^+ + 3w_1^- I\eta = 0$. Thus, $\mathrm{d}w_1^- = 3w_1^+ I\eta$ and $\mathrm{d}w_1^+ = -3w_1^- I\eta$. Moreover, $\mathrm{d}\alpha = 2(w_1^+\mathrm{d}w_1^+ + w_1^-\mathrm{d}w_1^-) = 0$. Since *M* is connected, if $\alpha \neq 0$ in some point, then $\alpha \neq 0$ everywhere. Now, it is immediate to check $3\alpha I\eta = w_1^+\mathrm{d}w_1^- - w_1^-\mathrm{d}w_1^+$ and $3\alpha\mathrm{d}I\eta = 2\mathrm{d}w_1^+\wedge\mathrm{d}w_1^- = 0$. Thus, parts (i) and (ii) of Theorem are already proved.

Parts (iii), (iv) and (v) are immediate consequences of parts (i) and (ii). □

Remark 3.8. In [7], Gray proved that if *M* is a connected nearly Kähler six-manifold (type $\mathcal{W}_1$) which is not Kähler, then *M* is an Einstein manifold such that $Ric = 5\alpha\langle\cdot,\cdot\rangle$, where

Ric denotes the Ricci curvature. In [12], showing an alternative proof of such Gray's result, we make use of Theorem 3.7.

3.2. Four dimensions

Now, let us pay lead our attention to manifolds with $SU(2)$-structure.

Theorem 3.9. *Let M be a special almost Hermitian four-manifold with Kähler form* ω *and complex volume form* $\Psi = \psi_+ + \mathrm{i}\psi_-$. *Then*

$$\nabla\psi_+ \in T^*M \otimes \omega + T^*M \otimes \psi_-, \qquad \nabla\psi_- \in T^*M \otimes \omega + T^*M \otimes \psi_+,$$

and $\Xi_\pm(\mathcal{W}_2) = \Xi_\pm(\mathcal{W}_4) = T^*M \otimes \omega$. *In this case, the space* $\mathcal{W} = \mathcal{W}_2 + \mathcal{W}_4$ *of covariant derivatives of* ω *also admits the relevant* $SU(2)$*-decomposition* $\mathcal{W} = T^*M \otimes \psi_+ + T^*M \otimes \psi_-$, *being ker* $\Xi_+ = T^*M \otimes \psi_-$ *and ker* $\Xi_- = T^*M \otimes \psi_+$.

If we consider the one-forms ξ_+ and ξ_- defined by $\nabla\omega = \xi_+ \otimes \psi_+ + \xi_- \otimes \psi_-$, i.e., $\xi_+ = \langle\nabla \cdot \omega, \psi_+\rangle$ and $\xi_- = \langle\nabla \cdot \omega, \psi_-\rangle$. The two decompositions of ξ are related as follows:

(i) $\xi \in \mathcal{W}_2$ if and only if $\xi_+ = I\xi_-$.
(ii) $\xi \in \mathcal{W}_4$ if and only if $\xi_+ = -I\xi_-$.

Moreover, we have the following consequences of last Theorem.

Corollary 3.10. *For* $SU(2)$*-structures, the exterior derivatives of* ψ_+, ψ_- *and* ω *are such that*

$$\begin{aligned}
\mathrm{d}\psi_+ &= -\xi_+ \wedge \omega - 2\eta \wedge \psi_+ = (\xi_+ \lrcorner \psi_- - 2\eta) \wedge \psi_+,\\
\mathrm{d}\psi_- &= -\xi_- \wedge \omega - 2\eta \wedge \psi_- = -(\xi_- \lrcorner \psi_+ + 2\eta) \wedge \psi_-,\\
\mathrm{d}\omega &= (\xi_+ - I\xi_-) \wedge \psi_+ = (\xi_+ \lrcorner \psi_- - \xi_- \lrcorner \psi_+) \wedge \omega.
\end{aligned}$$

Hence the one-forms ξ_+, ξ_- and η satisfy

$$-\xi_+ \lrcorner \psi_- + 2\eta = *(*\mathrm{d}\psi_+ \wedge \psi_+), \qquad \xi_- \lrcorner \psi_+ + 2\eta = *(*\mathrm{d}\psi_- \wedge \psi_-),$$
$$\xi_- \lrcorner \psi_+ - \xi_+ \lrcorner \psi_- = *(*\mathrm{d}\omega \wedge \omega).$$

Therefore, by Lemma 2.2, we have

$$\begin{aligned}
4\eta &= *(*\mathrm{d}\psi_+ \wedge \psi_+) + *(*\mathrm{d}\psi_- \wedge \psi_-) - *(*\mathrm{d}\omega \wedge \omega),\\
2\xi_- \lrcorner \psi_+ &= *(*\mathrm{d}\psi_- \wedge \psi_-) - *(*\mathrm{d}\psi_+ \wedge \psi_+) + *(*\mathrm{d}\omega \wedge \omega),\\
2\xi_+ \lrcorner \psi_- &= *(*\mathrm{d}\psi_- \wedge \psi_-) - *(*\mathrm{d}\psi_+ \wedge \psi_+) - *(*\mathrm{d}\omega \wedge \omega).
\end{aligned}$$

Thus we can conclude that all the information about an $SU(2)$-structure is contained in $\mathrm{d}\omega$, $\mathrm{d}\psi_+$ *and* $\mathrm{d}\psi_-$. Moreover, from these identities, the equalities for $n = 2$ contained in Theorem 2.6 follow.

By Proposition 2.12, for conformal changes of metric given by $\langle\cdot,\cdot\rangle_o = e^{2f}\langle\cdot,\cdot\rangle$, we have $I\mathrm{d}^*\omega_o = I\mathrm{d}^*\omega - 4\mathrm{d}f$ and $\eta_o = \eta - 1/2\mathrm{d}f$. The one-forms ξ_+ and ξ_- are modified in the way $\xi_{+o} = \xi_+ - \mathrm{d}f \lrcorner \psi_-$, $\xi_{-o} = \xi_- + \mathrm{d}f \lrcorner \psi_+$, where ξ_{+o} and ξ_{-o} are the respective one-forms corresponding to the metric $\langle\cdot,\cdot\rangle_o$. In fact, such identities can be deduced taking the expression $2\nabla_o\omega_o = e^{2f}\{2\nabla\omega - e_i \otimes e_i \wedge Idf - e_i \otimes Ie_i \wedge \mathrm{d}f\}$ into account, where ∇_o is the Levi-Civita connection of $\langle\cdot,\cdot\rangle_o$.

3.3. *Two dimensions*

Finally, let us consider special almost Hermitian two-manifolds. For these manifolds we have $\nabla\omega = 0$. Therefore,

$$\nabla\psi_+ = -I\eta \otimes \psi_- = -\eta_+\psi_- \otimes \psi_- + \eta_-\psi_+ \otimes \psi_- \in \mathbb{R} + \mathbb{R},$$
$$\nabla\psi_- = I\eta \otimes \psi_+ = \eta_+\psi_- \otimes \psi_+ - \eta_-\psi_+ \otimes \psi_+ \in \mathbb{R} + \mathbb{R},$$

where $\eta = \eta_+\psi_+ + \eta_-\psi_-$. Furthermore, $\mathrm{d}\psi_+ = -\eta_-\omega \in \mathbb{R}\omega$ and $\mathrm{d}\psi_- = \eta_+\omega \in \mathbb{R}\omega$. Consequently, $\eta_+ = - * \mathrm{d}\psi_-$ and $\eta_- = *\mathrm{d}\psi_+$.

With respect to the curvature, if K denotes the sectional curvature, it can be checked

$$K(\psi_+, \psi_-) = \mathrm{d}I\eta(\psi_+, \psi_-) = \mathrm{d}\eta_+(\psi_+) + \mathrm{d}\eta_-(\psi_-) - \eta_+^2 - \eta_-^2.$$

For conformal changes of metric given by $\langle\cdot,\cdot\rangle_o = e^{2f}\langle\cdot,\cdot\rangle$, the intrinsic $SU(1)$-torsion is modified in the way $e^f\eta_{+o} = \eta_+ - \mathrm{d}f(\psi_+)$ and $e^f\eta_{-o} = \eta_- - \mathrm{d}f(\psi_-)$, i.e., $\eta_o = \eta - \mathrm{d}f$.

Remark 3.11. Let us consider an special almost Hermitian $2n$-manifold, $n \geq 2$, which is Kähler (type $\mathcal{W}_5$). In such manifolds we have

$$\mathrm{d}\psi_+ = -n\eta \wedge \psi_+ = -nI\eta \wedge \psi_-, \qquad \mathrm{d}\psi_- = -n\eta \wedge \psi_- = nI\eta \wedge \psi_+.$$

By differentiating these identities, it follows $\mathrm{d}\eta \wedge \psi_+ = \mathrm{d}\eta \wedge \psi_- = 0$ and $\mathrm{d}I\eta \wedge \psi_+ = \mathrm{d}I\eta \wedge \psi_- = 0$. Therefore, $\mathrm{d}\eta, \mathrm{d}I\eta \in \mathfrak{su}(n) + \mathbb{R}\omega$.

4. Almost hyperhermitian geometry

A $4n$-dimensional manifold M is said to be *almost hyperhermitian*, if M is equipped with a Riemannian metric $\langle\cdot,\cdot\rangle$ and three almost complex structures I, J, K satisfying $I^2 = J^2 = -1$ and $K = IJ = -JI$, and $\langle AX, AY\rangle = \langle X, Y\rangle$, for all $X, Y \in T_xM$ and $A = I, J, K$. This is equivalent to saying that M has a reduction of its structure group to $Sp(n)$. As it was pointed out in Section 2, each fibre T_mM of the tangent bundle can be consider as complex vector space, denoted $T_mM_{\mathbb{C}}$, by defining $\mathrm{i}x = Ix$.

On $T_mM_{\mathbb{C}}$, there is an $Sp(n)$-invariant complex symplectic form $\varpi_{I\mathbb{C}} = \omega_J + \mathrm{i}\omega_K$ and a quaternionic structure map defined by $y \to Jy$. Taking our identification of $\overline{TM}_{\mathbb{C}}$ with $T^*M_{\mathbb{C}}$, $x \to \langle\cdot, x\rangle_{\mathbb{C}} = x_{\mathbb{C}}$, into account (we recall $\langle\cdot,\cdot\rangle_{\mathbb{C}} = \langle\cdot,\cdot\rangle + \mathrm{i}\omega_I(\cdot,\cdot)$), it is obtained

$\varpi_{I\mathbb{C}} = Je_{i\mathbb{C}} \wedge e_{i\mathbb{C}}$, where $e_1, \dots, e_n, Je_1, \cdots, Je_n$ is a unitary basis for vectors. Therefore,

$$\varpi^n_{I\mathbb{C}} = (-1)^{n(n+1)/2} n!\, e_{1\mathbb{C}} \wedge \cdots \wedge e_{n\mathbb{C}} \wedge Je_{1\mathbb{C}} \wedge \cdots \wedge Je_{n\mathbb{C}}.$$

Hence, we can fix $\Psi_I = \psi_{I+} + \mathrm{i}\psi_{I-}$, defined by $(-1)^{n(n+1)/2} n!\, \Psi_I = \varpi^n_{I\mathbb{C}}$, as complex volume form.

By cyclically permuting the rôles of I, J and K in the above considerations, we will obtain two more complex volume forms Ψ_J and Ψ_K. Thus, M is really equipped with three $SU(2n)$-structures, i.e., the almost complex structures I, J and K, the complex volume forms Ψ_I, Ψ_J, and Ψ_K and the common metric $\langle\cdot,\cdot\rangle$. We could say that M has a *special almost hyperhermitian* structure. Furthermore, we also have

$$(-1)^{n(n+1)/2}(n-1)!\, \mathrm{d}\Psi_I = (\mathrm{d}\omega_J + \mathrm{i}\mathrm{d}\omega_K) \wedge (\omega_J + \mathrm{i}\omega_K)^{n-1}.$$

Hence, we can compute $\mathrm{d}\psi_{I+}$ and $\mathrm{d}\psi_{I-}$ from $\mathrm{d}\omega_J$ and $\mathrm{d}\omega_K$. Likewise, making use of considerations contained in Sections 2 and 3, $\nabla\omega_I$ can be computed from $\mathrm{d}\omega_I$, $\mathrm{d}\psi_{I+}$ and $\mathrm{d}\psi_{I-}$. By a cyclic argument, the same happens for $\nabla\omega_J$ and $\nabla\omega_K$.

Theorem 4.1. *In an almost hyperhermitian manifold, the covariant derivatives* $\nabla\omega_I$, $\nabla\omega_J$ *and* $\nabla\omega_K$ *of the Kähler forms and the covariant derivative* $\nabla\Omega = 2\sum_{A=I,J,K} \omega_A \wedge \nabla\omega_A$ *are determined by the exterior derivatives* $\mathrm{d}\omega_I$, $\mathrm{d}\omega_J$ *and* $\mathrm{d}\omega_K$.

In other words, $\mathrm{d}\omega_I$, $\mathrm{d}\omega_J$ and $\mathrm{d}\omega_K$ contain all the information about the intrinsic torsion of an $Sp(n)$-structure and the intrinsic torsion, determined by $\nabla\Omega$ ([14,10]), of the underlying $Sp(n)Sp(1)$-structure. In relation with last Theorem, we recall Swann's result [14] that, for $4n \geq 12$, all the information about the covariant derivative $\nabla\Omega$ is contained in the exterior derivative $\mathrm{d}\Omega = 2\sum_{A=I,J,K} \omega_A \wedge \mathrm{d}\omega_A$. Furthermore, one of the consequences of previous Theorem is the Hitchin's result [9] that if the three Kähler forms ω_I, ω_J and ω_K of an almost hyperhermitian manifold are all closed, then they are covariant constant. Almost hyperhermitian manifolds with covariant constant Kähler forms are called *hyperkähler* manifolds. Such manifolds are Ricci-flat.

If the two almost Hermitian structures determined by I and J are locally conformal Kähler (type $\mathcal{W}_4$), then the one determined by K is also locally conformal Kähler [11]. Furthermore, in such a case, the three structures have common Lee form. We recall that the Lee form is defined by $\theta_A = -1/(2n-1) Ad * \omega_A$, $A = I, J, K$ [8]. Therefore, in such a situation we really have a *locally conformal hyperkähler* manifold. Let us compute the intrinsic torsion of the $SU(2n)_A$-structures, $A = I, J, K$. For $A = I$, we get

$$\mathrm{d}\Psi_I = \frac{1}{(-1)^{n(n+1)/2}(n-1)!}\theta \wedge (\omega_J + \mathrm{i}\omega_K)^n = n\theta \wedge \Psi_I,$$

where $\theta = \theta_I = \theta_J = \theta_K$. Therefore, $\mathrm{d}\psi_{I+} = n\theta \wedge \psi_{I+}$ and, by Theorem 2.6, we obtain that the $\mathcal{W}_5$-part of the torsion is determined by

$$\eta_I = \frac{1}{2n(2n-1)} I\mathrm{d}^*\omega_I = -\frac{1}{2n}\theta.$$

Proceeding in a similar way for J and K, we obtain $\eta_I = \eta_J = \eta_K$. Furthermore, note that the relevant one-form $2n(2n-1)\eta_I - I\mathrm{d}^*\omega_I$, given by Proposition 2.12, vanishes. In summary, we have the following result.

Theorem 4.2. *For a locally conformal hyperkähler manifold of dimension $4n$ and a non null Lee-form θ, the three $SU(2n)$-structures are of type $\mathcal{W}_4 + \mathcal{W}_5$. Moreover, the $\mathcal{W}_5$-part of each one of such structures is determined by the same one form $\eta = -1/2n\,\theta$.*

As consequences of this Theorem, we have some results relative to hyperkähler manifolds.

Corollary 4.3.

(i) *If the three $SU(2n)$-structures of an almost hyperhermitian $4n$-manifold are of type $\mathcal{W}_4$, then the manifold is hyperkähler.*
(ii) *For hyperkähler manifolds, the intrinsic torsion of each $SU(2n)$-structure vanishes.*

Remark 4.4. Special almost Hermitian manifolds with zero intrinsic torsion can be called *$SU(n)$-Kähler manifolds*. The metric of such manifolds is Ricci flat. Thus, Corollary 4.3 is an alternative proof of the Ricci flatness of the hyperkähler metrics.

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References

[1] M. Berger, Sur les groupes d' holonomie homogène des variétés à connexion affine et des vari,t,s riemanniennes, Bull. Soc. Math. France 83 (1955) 279–330.
[2] T. Bröcker, T. Tom Dieck, Representations of Compacts Lie Groups, Graduate Text in Math, vol. 98, Springer, 1985.
[3] S. Chiossi, S. Salamon, The intrinsic torsion of $SU(3)$ and G_2-structures, Differential Geometry, Valencia 2001, World Science Publishing, River Edge, NJ, 2002, pp. 115–133.
[4] R. Cleyton, A. Swann, Einstein metrics via intrinsic or parallel torsion, Math. Z. 247 (2004) 513–528.
[5] M. Falcitelli, A. Farinola, S. Salamon, Almost-Hermitian geometry, Differ. Geom. Appl. 4 (1994) 259–282.
[6] A. Gray, Minimal varieties and almost Hermitian submanifolds, Michigan Math. J. 12 (1965) 273–279.
[7] A. Gray, The structure of nearly Kähler manifolds, Math. Ann. 223 (1976) 233–248.
[8] A. Gray, L.M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mater. Pura Appl. 123 (4) (1980) 35–58.
[9] N.J. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. 55 (1987) 59–126.
[10] F. Martín Cabrera, Almost quaternion-Hermitian manifolds, Ann. Global Anal. Geom. 25 (2004) 277–301.
[11] F. Martín Cabrera, A. Swann, Almost Hermitian structures and quaternionic geometries, Differ. Geom. Appl. 21 (2004) 199–214.
[12] F. Martín Cabrera, A. Swann, Curvature of (Special) Almost Hermitian manifolds, eprint arXiv:math.DG/0501062, January 2005.
[13] S. Salamon, Riemannian geometry and holonomy groups, Pitman Research Notes in Math. Series, vol. 201, Longman, 1989.
[14] A. Swann, Aspects symplectiques de la géométrie quaternionique, C. R. Acad. Sci. Paris 308 (1989) 225–228.